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Compatible eulerian circuits in K_n^{**}

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Abstract

Let K_n^{**} be the complete symmetric digraph with a loop at each vertex. We say that two eulerian circuits of K_n^{**} are compatible if no pair of arcs of the digraph are consecutive in both eulerian circuits. We prove that there exist $\phi(n)$ pairwise compatible eulerian circuits in K_n^{**}, where ϕ denotes the Euler function, and we give an effective construction of these circuits.

1. Introduction

Let G be a finite simple eulerian graph, i.e. a connected graph such that each vertex has an even degree. An *Euler tour* of such graph is a closed walk that traverses each edge exactly once. Jackson shows in [3] that if G is an eulerian graph of minimum degree $2k$, then G has a set of $k-1$ Euler tours such that each pair of adjacent edges of G is in at most one tour of G. The $k-1$ was improved to k in [4]. These Euler tours are said to be *pairwise compatible*. In [5], Kotzig proposed the following conjecture.

Conjecture 1. The complete graph K_{2d+1} has a set of $2d-1$ pairwise compatible Euler tours.

This conjecture is still open.

In the case of digraphs the problem has been studied by Fleischner and Jackson [2]. For digraphs, we are dealing with directed Euler tours, called *eulerian circuits*. In [2], it is proved that every eulerian digraphs without loops of minimum degree $2k$ has a set of $\lfloor \frac{1}{2}k \rfloor$ pairwise compatible eulerian circuits. In this paper, we prove the following theorem.

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Theorem 1. *The complete symmetric digraph K_n^{**} with a loop at each vertex has a set of $\phi(n)$ pairwise compatible eulerian circuits, where ϕ is the Euler function.*

2. Definitions and notations

The graph we are dealing with here is K_n^{**} the complete symmetric digraph with n vertices and a loop at each vertex. The set of vertices (resp. arcs) of K_n^{**} is denoted by $V(K_n^{**})$ (resp. $A(K_n^{**})$). We consider that the set of vertices of K_n^{**} are the n elements of the commutative ring $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$, denoted by $\{0, \dots, n-1\}$. An arc of K_n^{**} with an *initial* vertex x and a *terminal* vertex y is denoted by (x, y), and is *labelled* by $y - x \in \mathbb{Z}/n\mathbb{Z}$. $(x, y) \in A(K_n^{**})$ and $(z, w) \in A(K_n^{**})$ are said to be *consecutive* if $y = z$. Two consecutive arcs of K_n^{**} form a *transition*. If (x_1, x_2) and (x_2, x_3) are consecutive arcs, the associated transition is labelled by the word of two letters $\alpha_1\alpha_2$, where $\alpha_1 = x_2 - x_1$ and $\alpha_2 = x_3 - x_2$ belong to $\mathbb{Z}/n\mathbb{Z}$. Moreover, we shall say that a transition formed by (x_1, x_2) and (x_2, x_3) *crosses* the vertex x_2. For example, the transition formed by the arcs $(0, 1)$ and $(1, 3)$ of K_5^{**} (Fig. 1) has label 12 and crosses the vertex 1.

A path P of length k of K_n^{**} is a sequence of k distinct arcs denoted by $P = (x_1, x_2, \dots, x_k, x_{k+1})$ where $x_i \in V(K_n^{**})$ $\forall i$, $1 \leqslant i \leqslant k$. A path $P = (x_1, \dots, x_{k+1})$ of K_n^{**} so that $x_1 = x_{k+1}$ is called a *circuit*. A circuit is said to be *eulerian* if it traverses each arc of K_n^{**} once and only once. The length (i.e. number of arcs) of an eulerian

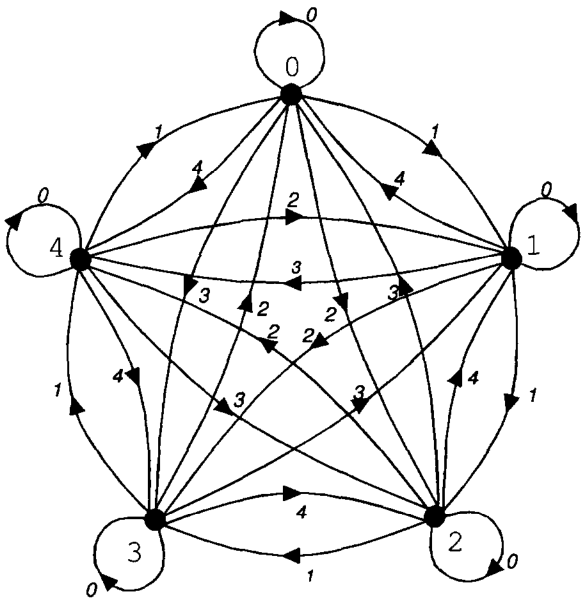

Fig. 1. K_5^{**} with labelled arcs.

circuit in K_n^{**} is equal to n^2. Two eulerian circuits of K_n^{**} are said to be *compatible* if no pair of arcs of K_n^{**} are consecutive in both eulerian circuits. We shall say that they do not use a same transition.

A path $P = (x_1, \ldots, x_{k+1})$ is also represented by a couple (x_1, S) where x_1 is the *initial vertex* and S is a word of k letters, $S = a_1 a_2 \ldots a_k$, where $a_i = x_{i+1} - x_i$ belongs to $\mathbb{Z}/n\mathbb{Z}$ for every i such that $1 \leqslant i \leqslant k$. So there is a bijection between the paths of length k having x_1 as initial vertex and the words of k letters belonging to $\{0, \ldots, n-1\}$. In what follows we do all computation using arithmetic modulo n; for more details, see for example [1]. Moreover, if $i \in \mathbb{Z}/n\mathbb{Z}$, we also shall denote by i the integer which is the unique representative of the class i belonging to $\{0, \ldots, n-1\} \subset \mathbb{N}$. Let a and b be two integers different from zero. We denote the greatest common divisor of a and b by $\gcd(a, b)$. Recall that a and b are *coprime* if and only if $\gcd(a, b) = 1$; for each $n \geqslant 1$, let $\phi(n)$ denotes the number of integer x in the range $1 \leqslant x < n$ such that x and n are coprime. The function ϕ is the Euler function, see [1, p. 49].

3. Compatible eulerian circuits of K_n^{**}

Our purpose is to build pairwise compatible eulerian circuits in K_n^{**}. To prove Theorem 1, we first give the following easy lemmas.

Lemma 1. *Let n be an integer different from* 0; *we have the following*:
- *If* $n \equiv 0 \pmod 2$, *then* $\sum_{i=1}^{n-1} i \equiv (n/2) \pmod n$.
- *If* $n \equiv 1 \pmod 2$, *then* $\sum_{i=1}^{n-1} i \equiv 0 \pmod n$.

Lemma 2. *Let n and i be two integers such that $1 \leqslant i \leqslant n$. Then* $\gcd(i, n) = 1$ *if and only if* $\gcd(n - i, n) = 1$.

Remark 1. Lemma 2 implies that for any integer n, $n \geqslant 1$, $\phi(n) \equiv 0 \pmod 2$.

Lemma 3. *Let n be an integer such that* $n \equiv 0 \pmod 4$. *For every integer i, if* $\gcd(n, i) = 1$, *then* $\gcd(n/2 - i, n) = 1$ *and* $\gcd(n/2 + i, n) = 1$.

Lemmas 2 and 3 imply the following result.

Lemma 4. *Let n be an integer, with $n > 4$. If* $n \equiv 0 \pmod 4$ *then* $\phi(n) \equiv 0 \pmod 4$.

We leave the proofs of these lemmas to the reader.

We now consider the array $\mathscr{A}$ with $n-1$ rows and $n-1$ columns defined as follows: $\forall i$, $\forall j$, $(i,j) \in \{1, \ldots, n-1\}^2$, $a_{ij} \in \{0, \ldots, n-1\}$ such that $a_{ij} \equiv i \times j \pmod n$, where $a_{i,j}$ is the element of the ith row and jth column of $\mathscr{A}$. We shall use the following lemma.

Lemma 5. *Let k and n be integers such that $1 \leqslant k \leqslant n-1$. If $\gcd(n,k)=1$ then*
- $a_{i,k} \neq a_{i',k}$ *with* $i \neq i'$, $(i,i') \in \{1,\dots,n-1\}^2$,
- $a_{k,j} \neq a_{k,j'}$ *with* $j \neq j'$, $(j,j') \in \{1,\dots,n-1\}^2$.

This lemma is an immediate consequence of the fact that if $\gcd(k,n)=1$ then the class of k in $\mathbb{Z}/n\mathbb{Z}$ is a generator of the abelian group $(\mathbb{Z}/n\mathbb{Z}, +)$.

In what follows, we consider the array $\mathscr{B}$ with $\phi(n)$ rows ($n>1$), and $n-1$ columns obtained from the array $\mathscr{A}$ by removing the rows of index k such that $\gcd(k,n) \neq 1$. The elements of $\mathscr{B}$ are denoted by $b_{i,j}$.

We can now give the construction of the pairwise compatible eulerian circuits in K_n^{**}; we consider three different cases.

3.1. $n \equiv 1 \pmod 2$

Let k be an integer such that $1 \leqslant k \leqslant \phi(n)$. We consider the following sequence:

$$S_k = (b_{k,1})^n (b_{k,2} \dots b_{k,n-1} 0)^n,$$

where $(ab\dots c)^k$ denotes k concatenations of the word $ab\dots c$. We denote by C_k the path $(0, S_k)$.

Claim 1. *C_k is an eulerian circuit of K_n^{**}.*

Proof. (1) According to the definition of S_k, it is easy to see that the last vertex of C_k is $x = 0 + n(\sum_{j=1}^{n-1} b_{k,j})$. Hence, $x \equiv 0 \pmod n$, and so C_k is a circuit. Moreover, C_k uses n^2 arcs.

(2) We show that all arcs in C_k are different. We recall that arcs with different labels are different. So, we only have to study arcs with identical labels in C_k.

We first consider the arcs labelled $b_{k,1}$. Since $\gcd(b_{k,1}, n) = 1$, $b_{k,1}$ is a generator of the abelian group $\mathbb{Z}/n\mathbb{Z}$. So, in C_k, by the definition of S_k, the arcs labelled $b_{k,1}$ have as initial vertices $0, b_{k,1}, \dots, (n-1)b_{k,1}$ which are all different.

We define now $\Lambda_k = \sum_{i=2}^{n-1} b_{k,i}$; then $\Lambda_k \equiv n - b_{k,1} \pmod n$ by Lemma 1, hence by Lemma 2, $\gcd(n, \Lambda_k) = 1$.

We consider the arcs labelled $b_{k,i}$, $i > 1$. Let γ be the initial vertex of the first traversed arc labelled by $b_{k,i}$, by going from 0 in the sense of the orientation in C_k. The arcs labelled by $b_{k,i}$ have as initial vertices $\gamma, \gamma + \Lambda_k, \dots, \gamma + (n-1)\Lambda_k$ which are all different (here again Λ_k is a generator of the abelian group $\mathbb{Z}/n\mathbb{Z}$). The same argument holds for the arcs labelled 0. So, the n^2 arcs in C_k are all different. This completes the proof. □

Claim 2. *Let k and k' be two integers such that $(k,k') \in \{1,\dots,\phi(n)\}^2$ and $k \neq k'$. Then the two eulerian circuits C_k and $C_{k'}$ are compatible.*

Proof. We assume that C_k and $C_{k'}$ use the same transition a_1a_2. In C_k, the label of a transition is one of the following five types:

type 1: $b_{k,1}b_{k,1}$,
type 2: $b_{k,j}b_{k,j+1}$,
type 3: $b_{k,n-1}0$,
type 4: $0b_{k,1}$,
type 5: $0b_{k,2}$.

We recall that $b_{k,j} \neq 0$ for $j \in \{1,\ldots,n-1\}$ and $b_{k,j+1} - b_{k,j} \equiv b_{k,1} \pmod{n}$. Moreover, two transitions crossing a given vertex are equal if and only if they have the same label. Hence, by definition and Lemma 5, it is clear that a_1a_2 cannot be of the same type in C_k and in $C_{k'}$. If they have different types, the only possibility is a_1a_2 of type 4 in C_k and of type 5 in $C_{k'}$, or conversely. This implies that $b_{k,1} = b_{k',2}$. On the other hand, there is only one transition labelled $0b_{k,1}$ (resp. $0b_{k',1}$) in C_k (resp. $C_{k'}$) and this transition crosses 0. The transitions labelled $0b_{k',2}$ cross the vertices

$$A_{k'}, 2A_{k'}, \ldots, (n-1)A_{k'}.$$

Since $\gcd(A_{k'}, n) = 1$, these vertices are different from 0. Hence, the transitions labelled $0b_{k',2}$ do not cross 0. Thus, a_1a_2 cannot have labels of these types. This completes the proof. □

3.2. $n \equiv 2 \pmod{4}$

In this case, $\forall k$, $1 \leqslant k \leqslant \phi(n)$, $\gcd(\sum_{j=2}^{n-1} b_{k,j}, n) > 1$. Since in the previous case, to prove that C_k is an eulerian circuit, we used the fact that $\gcd(\sum_{j=2}^{n-1} b_{k,j}, n) = 1$, we shall need another construction. So, we consider for $k \in \{1,\ldots,\phi(n)\}$ the sequence

$$S_k = [(b_{k,1})^{n/2}(b_{k,2}\ldots b_{k,n-1}0)^{n/2}]^2.$$

We denote by C_k the path $(0, S_k)$.

Lemma 6. $\forall (k,k') \in \{1,\ldots,\phi(n)\}^2$, $k \neq k'$, $b_{k,2} \neq b_{k',2}$.

Proof. We assume that for $k < k'$, $(k,k') \in \{1,\ldots,\phi(n)\}^2$, $b_{k,2} = b_{k',2}$. By construction $b_{k,1} = a$, $b_{k',1} = b$ with $a < b < n$. Because $\gcd(a,n) = 1$ and $\gcd(b,n) = 1$, a and b are odd. Moreover, $b_{k,2} \equiv 2a \pmod{n}$ and $b_{k',2} \equiv 2b \pmod{n}$, so $2(b-a) \equiv 0 \pmod{n}$, i.e. $b = a + n/2$. Since $n/2$ is odd, then a or b is even, which contradicts the hypothesis. □

Claim 3. *C_k is an eulerian circuit.*

Proof. As before, it is easy to see that the path C_k uses n^2 arcs and that it is a circuit. Now we show that all arcs in C_k are different. We only have to study the arcs with an identical label. We will use the following lemma.

Lemma 7. *Let $\Lambda_k = \sum_{i=2}^{n-1} b_{k,i}$. Then we have the following*:

(1) $\Lambda_k \equiv n/2 - b_{k,1} \pmod n$,

(2) $\Lambda_k \equiv 0 \pmod 2$,

(3) $(n/2)(\Lambda_k + b_{k,1}) \equiv (n/2)b_{k,1} \pmod n$,

(4) $(n/2)b_{k,1} \equiv (n/2) \pmod n$.

The proof of this lemma is left to the reader.

Proof of Claim 3 (*continued*). We consider the arcs labelled by $b_{k,1}$ in C_k. The arcs labelled by $b_{k,1}$ have the following initial vertices:

$$0, b_{k,1}, \ldots, \left(\frac{n}{2} - 1\right) b_{k,1}, \frac{n}{2} b_{k,1} + \Lambda_k \frac{n}{2}, \frac{n}{2} b_{k,1} + \Lambda_k \frac{n}{2} + b_{k,1}, \ldots,$$

$$\frac{n}{2} b_{k,1} + \Lambda_k \frac{n}{2} + \left(\frac{n}{2} - 1\right) b_{k,1}.$$

By Lemma 7(4), these vertices are

$$0, b_{k,1}, \ldots, \left(\frac{n}{2} - 1\right) b_{k,1}, \frac{n}{2} b_{k,1}, \ldots, (n-1) b_{k,1}.$$

Since $\gcd(b_{k,1}, n) = 1$ these vertices are all different.

We will now study the arcs labelled by $b_{k,j}$, $2 \leqslant j \leqslant n-1$. Let γ be the initial vertex of the first arc labelled by $b_{k,j}$ in C_k. Then the arcs labelled by $b_{k,j}$ in C_k have as initial vertices

$$\gamma, \gamma + \Lambda_k, \ldots, \gamma + \left(\frac{n}{2} - 1\right)\Lambda_k, \gamma + \frac{n}{2}\Lambda_k + \frac{n}{2} b_{k,1},$$

$$\gamma + \frac{n}{2}\Lambda_k + \frac{n}{2} b_{k,1} + \Lambda_k, \ldots, \gamma + \frac{n}{2}\Lambda_k + \frac{n}{2} b_{k,1} + \left(\frac{n}{2} - 1\right)\Lambda_k.$$

By Lemma 7(4), these vertices are the following: $\gamma, \gamma + \Lambda_k, \ldots, \gamma + (n/2 - 1)\Lambda_k$, $\gamma + n/2$, $\gamma + n/2 + \Lambda_k, \ldots, \gamma + n/2 + (n/2 - 1)\Lambda_k$. Moreover, if γ is even (resp. odd), $\gamma + i\Lambda_k$ $(1 \leqslant i \leqslant n/2 - 1)$ is even (resp. odd) and $\gamma + n/2 + j\Lambda_k$ $(0 \leqslant j \leqslant n/2 - 1)$ is odd (resp. even) because $n \equiv 2 \pmod 4$.

To prove that all these vertices are different, we prove that $i\Lambda_k \not\equiv j\Lambda_k \pmod n$ when $0 \leqslant i < j \leqslant n/2 - 1$. Let us assume that there exists a couple $(i,j) \in \{0, \ldots, n/2 - 1\}^2$, $i < j$, such that $i\Lambda_k \equiv j\Lambda_k \pmod n$, i.e. $(j - i)\Lambda_k \equiv 0 \pmod n$. Then

$$\left.\begin{array}{l} (j-i)\Lambda_k \equiv 0 \pmod n, \\ \gcd(\Lambda_k, n) = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} (j-i)\dfrac{\Lambda_k}{2} \equiv 0\left(\bmod \dfrac{n}{2}\right), \\ \left(\dfrac{\Lambda_k}{2}, \dfrac{n}{2}\right) = 1 \end{array}\right\} \Rightarrow (j = i)$$

is a contradiction. The same argument holds for the arcs labelled 0. This completes the proof. □

Claim 4. *Let k and k' be two integers such that $(k, k') \in \{1, \ldots, \phi(n)\}^2$ and $k \neq k'$. Then the two eulerian circuits C_k and $C_{k'}$ are compatible.*

Proof. We assume that C_k and $C_{k'}$ use the same transition labelled by $a_1 a_2$. As in the previous case by Lemmas 5 and 6, the only possibility is, w.l.o.g., that $a_1 a_2$ is of type 4 in C_k and type 5 in $C_{k'}$.

In C_k (resp. $C_{k'}$), there are only two transitions labelled $0b_{k,1}$ (resp. $0b_{k',1}$) crossing the two vertices 0 and $(n/2)b_{k,1}$ (resp. $(n/2)b_{k',1}$) by Lemma 7(3), which are in fact the vertices 0 and $n/2$ by Lemma 7(4).

The transitions labelled $0b_{k',2}$ cross the vertices $n/2 + \Lambda_{k'}, \ldots, n/2 + (n/2 - 1)\Lambda_{k'}$ and the vertices $\Lambda_{k'}, \ldots, (n/2 - 1)\Lambda_{k'}$. Those vertices are different from 0 and $(n/2)b_{k',1}$. Thus, the transitions labelled $0b_{k',2}$ do not cross these vertices. Hence, $a_1 a_2$ cannot have labels of these types. This completes the proof. □

3.3. $n \equiv 0 \pmod 4$

3.3.1. The case of $n \neq 4$

We first prove the following lemma.

Lemma 8. *Let k and k' be two integers $(k, k') \in \{1, \ldots, \phi(n)\}^2$ such that $k < k'$. Then we have the following:*

(1) $b_{k,2} = b_{k',2}$ if and only if $k' = k + \phi(n)/2$,

(2) $b_{k,n-2} = b_{k',n-2}$ if and only if $k' = k + \phi(n)/2$.

Proof. (1) We consider the array $\mathscr{A}$. W.l.o.g., let i and i' be two integers such that $1 \leqslant i < i' \leqslant n - 1$. It is easy to see that $a_{i',2} \equiv a_{i,2} \pmod n$ if and only if $i' = i + n/2$. Let now k and k' be two integers such that $1 \leqslant k < k' \leqslant \phi(n)$ and i and i' such that $b_{k,1} = i$, $b_{k',1} = i'$. We have

$$b_{k',2} = b_{k,2} \Leftrightarrow 2i = 2i' \pmod n \Leftrightarrow i' = i + \frac{n}{2}.$$

Since by Lemma 2, $|\{q \in \{1, \ldots, n/2 - 1\}\colon \gcd(q, n) = 1\}| = |\{q \in \{n/2 + 1, \ldots, n - 1\}\colon \gcd(q, n) = 1\}|$, by Lemma 3 we have $b_{k',2} = b_{k,2} \Leftrightarrow k' = k + \phi(n)/2$.

(2) It can be proved by using similar arguments. □

Remark 2. Let k and k' be two integers such that $1 \leqslant k < k' \leqslant \phi(n)$. It is easy to see that, by Lemma 2, we have $b_{k',1} = n - b_{k,1} \Leftrightarrow k' = \phi(n) + 1 - k$.

Now we deal with a new array $\mathscr{B}^s$ by applying the following algorithm.

Algorithm

```
for k = 1 to φ(n)/2 do
  if k is even then
    for i ∈ {k, φ(n) + 1 − k} do
      begin
        b^s_{i,1} = b_{i,n−1}
        for j = 2 to n − 1 do
          b^s_{i,j} = b_{i,j−1}.
      end.
```

Lemma 9. *For every* $(k,k')\in\{1,\dots,\phi(n)\}^2$, $k\neq k'$, *we have the following*:

(1) $b^s_{k,1}\neq b^s_{k',1}$,

(2) $b^s_{k,2}\neq b^s_{k',2}$,

(3) $b^s_{k,n-1}\neq b^s_{k',n-1}$.

Proof. (1) By Lemma 5, if $(k,k')\in\{1,\dots,\phi(n)\}^2$, $k\neq k'$, we know that $b_{k,1}\neq b_{k',1}$ and $b_{k,n-1}\neq b_{k',n-1}$. So, we only have to compare two elements $b^s_{k,1}$ and $b^s_{k',1}$ such that $b^s_{k,1}=b_{k,n-1}$ and $b^s_{k',1}=b_{k',1}$; hence k is even. If $b^s_{k',1}=b^s_{k,1}$, we have $b_{k',1}=n-b_{k,1}$ which is equivalent to $k'=\phi(n)-1+k$. By the definition of $\mathscr{B}^s$, we have $b^s_{k',1}=b_{k',n-1}$, a contradiction.

(2) For $k\in\{1,\dots,\phi(n)\}$, if $b^s_{k,2}=b_{k,2}$, then $b^s_{k,2}$ is even, and if $b^s_{k,2}=b_{k,1}$ then $b^s_{k,2}$ is odd. Hence, we study the set of even values and the set of odd values of the second column of $\mathscr{B}^s$.

- *The set of even values.* Let k and k' be two integers such that $1\leqslant k<k'\leqslant\phi(n)$, $b^s_{k,2}=b^s_{k',2}$, $b^s_{k,2}=b_{k,2}$ and $b^s_{k',2}=b_{k',2}$. Then we have $b_{k,2}=b_{k',2}$, so by Lemma 8(1), $k'=k+\phi(n)/2$. Since $b^s_{k,2}=b_{k,2}$ and $k<\phi(n)/2$, then k is odd. Hence, by Lemma 4, k' is odd. Since $k'>\phi(n)/2$ we deduce from the definition of $\mathscr{B}^s$ that $b^s_{k',2}=b_{k',1}$, which implies $b_{k',1}=0$, a contradiction.
- *The set of odd values.* By the definition of $\mathscr{B}$, it is clear that $b_{k,1}\neq b_{k',1}$ for every $(k,k')\in\{1,\dots,\phi(n)\}^2$ and $k\neq k'$.

(3) By using Lemma 8(2), it is easy to prove that $b^s_{k,n-1}\neq b^s_{k',n-1}$.

This completes the proof of the lemma. □

Now we need a last lemma to complete the proof of the theorem.

Lemma 10. *Let* $\Lambda_k=\sum_{i=2}^{n-1}b^s_{k,i}$ *for* $k\in\{1,\dots,\phi(n)\}$. *Then* $\gcd(\Lambda_k,n)=1$.

Proof. $\Lambda_k=\sum_{i=2}^{n-1}b^s_{k,i}$ have two possible values:

- If $b^s_{k,1}=b_{k,1}$, by Lemma 1, $\Lambda_k\equiv n/2-b_{k,1}\pmod n$.
- If $b^s_{k,1}=b_{k,n-1}$, by Lemma 1, $\Lambda_k\equiv n/2-(n-b_{k,1})\pmod n$, i.e. $\Lambda_k\equiv n/2+b_{k,1}\pmod n$.

In both cases Lemma 3 implies $\gcd(\Lambda_k,n)=1$. This completes the proof. □

These two lemmas prove that the array $\mathscr{B}^s$ has similar properties as the ones of $\mathscr{B}$ which we use in the case of n odd. Hence, by considering the word

$$S_k = (b^s_{k,1})^n (b^s_{k,2} \dots b^s_{k,n-1} 0)^n$$

for every $k \in \{1, \dots, \phi(n)\}$, it is easy to prove with the same arguments that K_n^{**} has $\phi(n)$ pairwise compatible eulerian circuits.

3.3.2. The case of $n = 4$

For $n = 4$, we have the following result.

Lemma 11. *There exist three pairwise compatible eulerian circuits of K_4^{**}.*

Proof. The three pairwise compatible eulerian circuits we propose are the following:

- $C_1 = 0033231302212011$.
- $C_2 = 0010203112132233$.
- $C_3 = 0023211331030122$. □

This completes the proof of Theorem 1.

4. Conclusion

(1) The following result is an easy corollary of a theorem due to Fleischner and Jackson [2].

In the complete symmetric digraph with n vertices, K_n^*, there exist $\lfloor (n-1)/2 \rfloor$ compatible eulerian circuits.

By suitable adding the loops of K_n^{**} to the pairwise compatible eulerian circuits of K_n^*, we can show that there exist $\lfloor (n-1)/2 \rfloor$ compatible eulerian circuits in K_n^{**}. This result improves for some values of n the one given by our main theorem.

(2) Lemma 11 gives three pairwise compatible eulerian circuits for $n = 4$.

In fact we propose the following conjecture.

Conjecture 2. K_n^{**} has a set of $n - 1$ pairwise compatible eulerian circuits.

By our result this conjecture is verified for n prime.

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